UDC 531.36:534

THE RING PRINCIPLE IN PROBLEMS OF INTERACTION BETWEEN TWO SELF-OSCILLATING SYSTEMS

PMM Vol. 41, №4, 1977, pp. 618-627

V. S. AFRAIMOVICH and L. P. SHIL'NIKOV
(Gor'kii)

(Received July 12, 1976)

It is shown that an attracting set containing a denumerable set of periodic motions may exist in the presence of two interacting self-oscillating systems.

Presence of synchronisation and beat modes under the action of external periodic perturbation on a self-oscillating system was established in the works of Van der Pol, Andronov, and A. A. Vitte. In nonresonant cases the problem of existence of beat modes can be reduced to that of existence of stable invariant tori. It was shown by Krylov and Bogoliubov [1] that with a suitable selection of the secant, the existence of an invariant torus is implied by the existence of an invariant curve in the mapping of a ring into a ring. Further investigation of problems of existence and smoothness of periodic surfaces are treated in the works of Bogoliubov, Iu. A. Mitropolskii, Levinson, Diliberto, Heil, and others.

1. It can happen that an invariant curve does not exist for mapping a ring into a ring, and that only a closed invariant set of a complex nature which contains a denumerable set of periodic motions, a continuum of trajectories that are Poisson stable, etc. is available. Below we present a mathematical description of such situation, formulated in the form of the principle of ring (for simplicity the exposition here is in a less general form than in [2]).

The principle of ring. Let the mapping T

$$\bar{x} = f(x, \theta), \ \bar{\theta} = \theta + g_1(x, \theta) \equiv g(x, \theta) \ (\mathrm{mod}\ 2\pi)$$
$$x = (x_1, \ldots, x_k)$$

where f and g_1 are C^1-smooth functions 2π-periodic with respect to θ, maps ring $K: \| x \| \leqslant r_0, \ 0 < \theta \leqslant 2\pi$ into itself and satisfies the following conditions:

1°. $\| f_x \| \leqslant q < 1$ for all (x, θ);

2°. The condition of phase dilatation is satisfied, i.e. there exist a $p > 1$ and a segment $[a, b]$, $0 < a < b \leqslant 2\pi$, such that $\max_x | g_\theta^{-1} | \leqslant p^{-1}$ for any $\theta \in [a, b]$. Moreover $| g(x, b) - g(x, a) | > 2\pi(n + 1)$, $n \geqslant 2$ for any $x, \| x \| \leqslant r_0$;

3°. $\| f_\theta \| \, \| g_x \| < (p - 1)(1 - q)$ for all $(x, \theta) \in K$.

There exists then in ring K a closed set Σ consisting of trajectories of T-mapping which can be set in a one-to-one and bicontinuous correspondence with the set of all possible sequencies that consist of n symbols and are infinite in both directions. In other words, mapping T onto Σ is topologically associated with the shift of the Bernoulli topological scheme of n symbols (see Fig. 1, where the shaded region represents image of K in the T - mapping).

Condition 2° means that the length of images of circles $x = x_0$ is increased not less than $(n + 1)$ times, which results in exponential scatter of points on the phase in intervals where $|g_\theta| > 1$ (see Fig. 1). The closeness of condition 2° to the heuristic stochasticity criterion of Chirikov indicated by him for area retaining mapping should be noted in this connection(*).

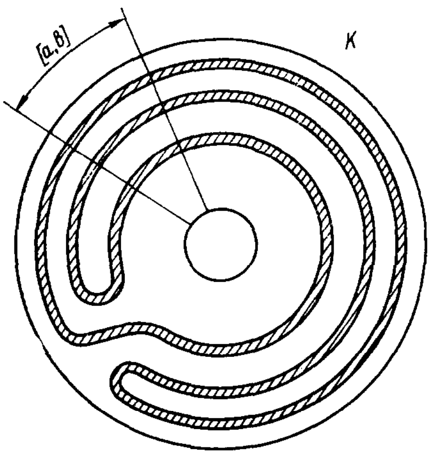

Fig. 1

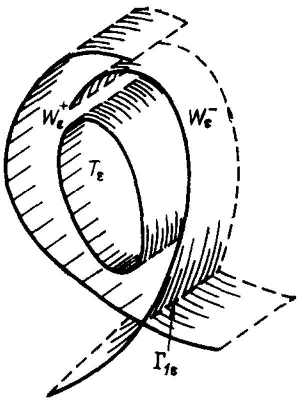

Fig. 2

Such cases were, evidently, not encountered in problems in which perturbations of input equations were fairly small. To indicate a system in which the principle of ring is realized it is necessary for either the interaction bond of the self-oscillating systems (or the perturbation of a self-oscillating system) to be fairly considerable, or for the degree of coarseness (i.e. the distance to the bifurcation boundary) of one of the systems to be small. The first course is very complicated and necessitates the use of a computer in specific cases. The second course permits the establishment of conditions in which the principle of ring can be applied. The result established in the present work is notably such that in the interaction of two self-oscillating systems, one of which has a limit cycle which passes near the equilibrium state of the saddle type (that corresponds to the self-oscillation mode), the image of some secant into itself can satisfy the conditions of the principle of ring. The complex structure of trajectories for periodically perturbed two-dimensional self-oscillating systems was established in [2] with the use of the principle of ring. Other mechanics of formation of attracting sets of complex nature are indicated in [3, 4].

---

*) Chirikov, B. V., Investigation in the theory of nonlinear resonance and stochasticity. Preprint Ins. of Nuclear Phys. SO Akad. Nauk SSSR, 267, Novosibirsk, 1969.

2. Let us consider the system of differential equations

$$u^{\cdot} = U(u, \varepsilon), \quad \| u \| \leqslant \text{const}, \; \varepsilon \in [-\varepsilon_0, \varepsilon_0] \tag{2.1}$$

where u is the m-vector and the right-hand sides of the system are C^4-smooth vector functions of variables, and are C^3-smooth with respect to ε. We assume that system (2.1) has an equilibrium state O_ε which is a simple saddle for which the roots of the characteristic equation are $\lambda_1(\varepsilon), \ldots, \lambda_{m-1}(\varepsilon), \gamma(\varepsilon)$ and $\mathrm{Re}\lambda_i(\varepsilon) < 0$, $\gamma(\varepsilon) > 0$.

We assume that when $\varepsilon = 0$ the following conditions are satisfied:

1) there exists trajectory Γ_0 which is doubly asymptotic to the saddle O_0;

2) the saddle quantity is negative, i.e.

$$\max_i \{\mathrm{Re}\lambda_i(0)\} + \gamma(0) < 0 \tag{2.2}$$

If for $\varepsilon > 0$ the loop "collapses inwards", then, according to [5, 6], system (2.1) has a stable limit cycle Γ_ε. The degree of coarseness (i.e. the distance to the bifurcation surface which corresponds to the separatrix loop) of system (2.1) is evidently of order ε.

In case 1) system (2.1) is subjected to a small periodic perturbation of order μ: $u^{\cdot} = U(u, \varepsilon) + \mu U_1(t, u, \varepsilon, \mu)$, while in case 2) system (2.1) interacts with another self-oscillating system, i.e.

$$u^{\cdot} = U(u, \varepsilon) + \mu P(u, v, \mu) \tag{2.3}$$

$$v^{\cdot} = V(v) + \mu Q(u, v, \mu) \tag{2.4}$$

$$\| u \| \leqslant \text{const}, \quad \| v \| \leqslant \text{const}, \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0], \quad \mu \in [-\mu_0, \mu_0]$$

where v is an n-vector and the right-hand sides of system (2.3), (2.4) are C^4 smooth vector functions of variables and parameters ε and μ. We shall consider case 2), since the analysis of case 1) is analogous.

We assume that system (2.4) has a periodic motion $\Gamma_1 : v = v(\theta)$, $\theta = \omega t$, $\omega > 0$ when $\mu = 0$, which corresponds to a coarse limit cycle whose multiplicators are nonnegative. In a fairly small neighborhood Γ_1 with $\mu = 0$ system (2.4) can, according to the Floquet - Liapunov theory, be reduced by the C^4-smooth substitution of variables $v \to (r, \theta)$ to the form

$$r = Ar + B(r, \theta), \quad \theta^{\cdot} = \omega + C(r, \theta)$$

where r is the $(n-1)$-vector, B and C are 2π-periodic with respect to θ and C^3-smooth and the eigenvales $\alpha_1, \ldots, \alpha_{n-1}$ of matrix A are such that $\mathrm{Re}\,\alpha_i < 0$, $i = 1, \ldots, n-1$.

The assumptions made about systems (2.3) and (2.4) imply that when $\mu = 0$, that system for $\mu = 0$ and $\varepsilon > 0$ has a saddle periodic motion $\Gamma_{1\varepsilon} \equiv O_\varepsilon \times \Gamma_1$, whose stable manifold W_ε^+ is $(m + n - 1)$-dimensional, and the unstable W_ε is two-dimensional and, also, a two-dimensional invariant torus $T_\varepsilon \equiv \Gamma_\varepsilon \times \Gamma_1$. The latter passes for small ε near the periodic motion $\Gamma_{1\varepsilon}$ and, when $\varepsilon \to 0$ merges with the loop of the unstable manifold W_ε equal to $\Gamma_0 \times \Gamma_1$ (see Fig. 2, where $\mu = 0$ and $\varepsilon > 0$ and the solid lines show the traces of intersection of torus T_ε and manifolds W_ε^+ and W_ε^- with the secant $\theta = \text{const}$).

Let us assume that the saddle quantity of periodic motion is negative, i.e.

$$\max\{\operatorname{Re}\lambda_j(0),\ \operatorname{Re}\alpha_i\} + \gamma(0) < 0 \tag{2.5}$$

Note that in case 1) condition (2.5) is automatically satisfied when inequalities (2.2) are satisfied.

The analysis of behavior of trajectories of system (2.3), (2.4) in some small neighborhood of the torus T_ε is conveniently reduced to that of mapping the secant into itself on trajectories of system (2.3), (2.4). The secant is selected in the form of direct product of some secant to Γ_ε (and to Γ_0) in the small neighborhood Γ_1.

3. By a C^3-smooth nondegenerate substitution of variables and time we reduce system (2.3), (2.4) in the neighborhood $\Gamma_{1\varepsilon}$ to the form

$$x^{\cdot} = A(\varepsilon,\mu)x + F(x, y, \theta, \varepsilon, \mu)x, \quad y^{\cdot} = \gamma(\varepsilon,\mu)y \tag{3.1}$$
$$\theta = \omega + H(x, y, \theta, \varepsilon, \mu), \quad H = H_1 x + H_2 y$$

where x is the $(m+n-2)$-vector; y and θ are scalars, the right-hand sides of the system belong to class C^2 in region $G \equiv B_1 \times [-\varepsilon_1, \varepsilon_1] \times [-\mu_1, \mu_1]$ $(0 < \varepsilon_1 \leqslant \varepsilon_0,\ 0 < \mu_1 \leqslant \mu_0$ and B_1 is the δ-neighborhood of $\Gamma_{1\varepsilon})$; F and H are 2π-periodic with respect to θ and vanish when $x = y = 0$; eigenvalues of the $(m+n-2)\times(m+n-2)$-matrix $A(\varepsilon,\mu)$ are close to $\lambda_1(0), \ldots, \lambda_{m-1}(0)$ and $\alpha_1, \ldots, \alpha_{n-1}$, respectively, and the quantity $\gamma(\varepsilon,\mu)$ is close to $\gamma(0)$.

Note 1. The system can be obtained in form (3.1) by straightening the smooth invariant manifolds W_ε^+ and W_ε of periodic motion $\Gamma_{1\varepsilon}$ which by the Theorems 3-6 in [7] exist, by the substitution of time, and in accordance with the Floquet-Liapunov theory. It can also be shown that when $\mu = 0$, the combined substitution of (x, y, θ) for (u, v), y as a function of u and v is independent of v, and θ is independent of u.

For fairly small δ in G_1 we have the following estimates:

$$\|F_x\| + \|F_y\| + \|H_x\| + \|H_y\| \leqslant M, \quad \|F\| + \|H\| \leqslant M\delta \tag{3.2}$$
$$\|F_\theta\| + \|H_\theta\| < N\delta, \quad \|H_{1\theta}\| + \|H_{2\tilde{\theta}}\| < M'$$

where M, M, and N are constants that are independent of δ.

We seek transformation T in the form of superposition of mapping T_0 constructed by trajectories of system (2.3), (2.4) which pass in the neighborhood of $\Gamma_{1\varepsilon}$, and mapping T_1 by trajectories which pass in the neighborhood of $\Gamma_0 \times \Gamma_1$.

Mapping T_0. In conformity with assumptions the intersection of W_ε with the δ neighborhood of $\Gamma_{1\varepsilon}$ has for $\mu = \varepsilon = 0$ two connectedness components: W^{--} that belongs to cylinder $x = 0$ and W^{-+} that belongs to cylinder $y = 0$. Let us fix the circle P^-: $x = 0$, $y = d_1$, $0 < \theta \leqslant 2\pi$ on W^{--} and consider the secant S_1: $y = d_1$, $\|x\| \leqslant \rho_1$, $0 < \theta \leqslant 2\pi$. We also fix the circle P^+: $x = x^0$, $y = 0$, $0 < \theta \leqslant 2\pi$ on W^{-+} and select an arbitrary transversal to W^{-+} that intersects S_0 so that $\|x - x^\circ\| \leqslant \rho_0$, $\|y\| \leqslant \rho_0$ for points $(x, y, \theta) \in S_0$. For fairly small μ, ε, ρ_0 and ρ_1 the secants S_1 and S_0 are, obviously, transversal to the trajectories of system (2.3), (2.4).

Let

$$x = x(t, x_0, y_0, \theta_0), \quad \theta = \theta(t, x_0, y_0, \theta_0), \quad y = y_0 e^{\gamma(\varepsilon,\mu)t} \tag{3.3}$$

be the solution of system (3.1) that at $t = 0$ passes through the point (x_0, y_0, θ_0) and lies in the δ-neighborhood of $\Gamma_{1\varepsilon}$. We fix λ_1, $\max_{i,j}\{\operatorname{Re}\lambda_j(0), \operatorname{Re}\alpha_i\} < \lambda_1 < 0$ and γ_1, $\gamma_1 > \gamma(0) > 0$ such that $\lambda_1 + \gamma_1 < 0$ (they exist by virtue of assumption (2.5). It can be shown that the estimate

$$\| \exp A(\varepsilon, \mu) t \| \leqslant \text{const}\, e^{\lambda_1 t} \tag{3.4}$$

holds for all reasonably small ε and μ.

The following lemma is proved by the methods presented in [6, 8], with the use of inequalities (3.2) and (3.4) and the generalized Gronuall inequality.

Lemma 1. For any numbers α and β such that $\alpha > 0$, $\lambda_1 < \beta < 0$ there exist such numbers δ_2, μ_2 $(0 < \mu_2 \leqslant \mu_1)$, ε_2 $(0 < \varepsilon_2 \leqslant \varepsilon_1)$, that in region $G \equiv B_2 \times [-\varepsilon_2, \varepsilon_2] \times [-\mu_2, \mu_2]$, where B_2 is the δ-neighborhood of $\Gamma_{1\varepsilon}$ $(0 < \delta \leqslant \delta_2)$, estimates

$$\| x(t) \| \leqslant 2\delta e^{\lambda_1 t} \tag{3.5}$$

$$\left\| \frac{\partial x(t)}{\partial x_0} \right\| \leqslant C_1 e^{\beta t}, \quad \left\| \frac{\partial \theta(t)}{\partial x_0} \right\| \leqslant C_1' e^{N\delta t} \tag{3.6}$$

$$\left\| \frac{\partial x(t)}{\partial y_0} \right\| \leqslant C_2 e^{(\lambda_1+\gamma_1)t}, \quad \left\| \frac{\partial \theta(t)}{\partial y_0} \right\| \leqslant C_2' e^{\gamma(\varepsilon,\mu)t} \tag{3.7}$$

$$\left\| \frac{\partial x(t)}{\partial \theta_0} \right\| \leqslant C_3 \delta e^{(\lambda_1+\alpha)t}, \quad \left| \frac{\partial \theta(t)}{\partial \theta_0} - 1 \right| \leqslant C_3' \delta \tag{3.8}$$

where C_i and C_i' are constants, hold for the solution of Eqs. (3.1).

We select δ_2, ε_2, and μ_2 so small that α and β satisfy conditions

$$0 < \alpha < \gamma_1, \quad 0 < \alpha < -(\lambda_1 + \gamma_1), \quad \lambda_1 < \beta < \lambda_1 + \gamma_1 \tag{3.9}$$

and, furthermore, $C_3'\delta_2 < 1/8$, $M\delta_2 < \min\{1/8, 1/8\,\omega\}$.

These inequalities with allowance for (3.5) make it possible to obtain on the basis of the form of (3.1) the estimates

$$\left\| \frac{\partial x(t)}{\partial t} \right\| < 2\left(\|A(\varepsilon,\mu)\| + 1/8\right)\delta_2 e^{\lambda_1 t} \tag{3.10}$$

$$\left| \frac{\partial \theta(t)}{\partial t} - \omega \right| < \frac{1}{8}\omega$$

and the last of estimates (3.8) can now be written as

$$\left| \frac{\partial \theta(t)}{\partial \theta_0} - 1 \right| < \frac{1}{8} \tag{3.11}$$

The mapping $T_0\colon S_0 \to S_1$ is obtained by substituting into (3.3) the transition time

$$t_n = -\frac{1}{\gamma(\varepsilon,\mu)} \ln \frac{y_0}{d_1} \tag{3.12}$$

from S_0 to S_1, which is determined by the condition $d_1 = y_0 e^{\gamma(\varepsilon,\mu)t}$. We have

$$T_0\colon \bar{x}_1 = x\left(-\frac{1}{\gamma(\varepsilon,\mu)} \ln \frac{y_0}{d_1},\, x_0, y_0, \theta_0, \varepsilon, \mu\right) \tag{3.13}$$

$$\bar{\theta}_1 = \theta\left(-\frac{1}{\gamma(\varepsilon,\mu)}\ln\frac{y_0}{d_1},\ x_0,\ y_0,\ \theta_0,\ \varepsilon,\ \mu\right)$$

Inequalities (3.5) assume the form

$$\|x_1\| \leqslant 2\delta\left(\frac{y_0}{d_1}\right)^{\zeta}, \quad \zeta = -\frac{\lambda_1}{\gamma_1} \tag{3.14}$$

From this follows the statement: there exists for any $\rho_1 > 0$ such $b > 0$ that for all reasonably small ε_2 and μ_2 the mapping $T_0 : S_0 \to S_1$ is determined in region $\sigma_0 = \{(x, y, \theta) \in S_0,\ 0 < y \leqslant b\}$.

M a p p i n g T_1. By virtue of assumptions made above the circle P is transformed during a finite time into circle $x = x_0^\circ,\ y = 0,\ 0 < \theta \leqslant 2\pi$, on the trajectories of system (2.3), (2.4) when $\mu = \varepsilon = 0$. We take that circle as the circle P, i.e. we set $x^\circ = x_0^\circ$.

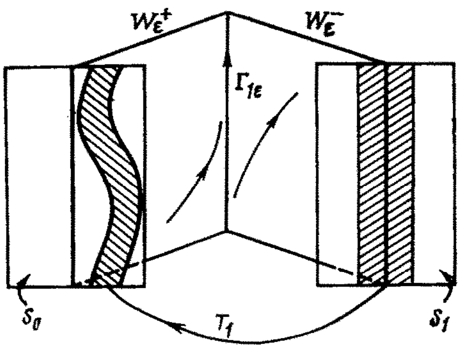

Fig. 3

It follows from general theorems that for all reasonably small ε and μ there exists a smooth nondegenerate image of the neighborhood of one of these circles onto some neighborhood of the other, i.e. for any $\rho_0 > 0$ there exists a $\rho_1 > 0$ such that a smooth nondegenerate mapping $T_1 : S_1 \to S_0$ exists.

When $\mu = \varepsilon = 0$ the system (2.3), (2.4) splits into systems (2.3) and (2.4), hence, owing to the initial condition $x_1 = 0$ the motion in system (2.4) occurs on the limit cycle $\Gamma_1 : v = v(\theta),\ \theta^\cdot = \omega$, and consequently, $\theta_0 = \theta_1 + \omega\tau_0$. This and the nondegeneracy and smoothness of the substitution of (x, y, θ) for (u, v) with respect to variables and parameters imply that for $\mu = k\varepsilon$ mapping T_1 can be defined by

$$\begin{aligned} x_0 &= x_0^\circ + \varepsilon P'_1(x_1, \theta_1, \varepsilon) + P_2'(x_1, \theta_1, \varepsilon) \\ y_0 &= \Delta_1(x_1, \theta_1, \varepsilon)\, x_1 + \varepsilon\Delta_2(x_1, \theta_1, \varepsilon) \\ \theta_0 &= \{\theta_1 + \omega\tau_0 + R_1'(x_1, \theta_1, \varepsilon)\, x_1 + \varepsilon R_2'(x_1, \theta_1, \varepsilon)\} \pmod{2\pi} \end{aligned} \tag{3.15}$$

where the right-hand sides belong to class C^2 and P_i, Δ_i and Q_i are 2π-periodic with respect to θ_1, $i = 1, 2$. By virtue of Note 2, $\Delta_2 \equiv R_1(x_1, \varepsilon) + kR_2(x_1, \theta_1, \varepsilon)$, and, if the case in which conditions of creation are satisfied, $R_1(x_1, \varepsilon) > 0$. Let us assume that $\Delta_2(0, \theta_1, 0) > 0$ (if Δ_2 vanishes it means that system (2.3), has homocline curves [8, 9] if, however, $\Delta_2 < 0$, all trajectories, except $\Gamma_{1\varepsilon}$, leave the neighborhood $\overline{\Gamma_0 \times \Gamma_1}$ [2].)

Stable and unstable manifolds of periodic motion $\Gamma_{1\varepsilon}$ are shown in Fig. 3 for $\varepsilon > 0$ and $\Delta_2 > 0$.

4. The mapping T is obtained as the superposition of mappings T_0 and T_1 by substituting (3.15) into (3.13) with the transition time t_n determined by the equality

$$t_n = -\frac{1}{\gamma(\varepsilon, k\varepsilon)} \ln \frac{y_0}{d_1}, \quad y_0 = \Delta_1(x_1, \theta_1, \varepsilon) x_1 + \varepsilon \Delta_2(x_1, \theta_1, \varepsilon) \qquad (4.1)$$

Let us show that the mapping T has actually the same properties as the model mapping

$$\bar{x}_1 = \frac{x_0^{\circ}}{d_1} y_0^{\zeta} \ (\approx e^{\lambda_1 t_n} \overset{\circ}{x_0})$$
$$\bar{\theta}_1 = \theta_1 + \omega\tau_0 + \frac{\omega}{\gamma(\varepsilon, k\varepsilon)} \ln \frac{y_0}{d_1} \quad (\approx \theta_1 + \omega\tau_0 + \omega t_n)$$

by proving the following lemma.

Lemma 2. For any $\varkappa$, $1 < \varkappa < \zeta$ there exist such ε_3 and δ_3 that for all ε, $0 < \varepsilon \leqslant \varepsilon_3$ mapping T transforms ring $K_\varepsilon : \| x_1 \| \leqslant \varepsilon^\varkappa$, $0 < \theta_1 \leqslant 2\pi$ into itself, and the estimates

$$\left\| \frac{\partial \bar{x}_1}{\partial x_1} \right\| \leqslant D_1 \varepsilon^{\zeta - 1}, \quad \left\| \frac{\partial \bar{x}_1}{\partial \theta_1} \right\| \leqslant D_2 \varepsilon^{\zeta - \alpha/\gamma_1} \qquad (4.2)$$
$$\left\| \frac{\partial \bar{\theta}_1}{\partial x_1} \right\| \leqslant D_3 \varepsilon^{-1}, \quad \left| \frac{\partial \bar{\theta}_1}{\partial \theta_1} \right| > \frac{6\omega}{8\gamma_1} \left| \frac{\Delta_{2\theta_1}(0, \theta_1, 0)}{\Delta_2(0, \theta_1, 0)} \right| - \frac{9}{8}$$

then hold.

Proof. 1°. Estimate (3.14), the second of formulas (3.15), and the inequality $\| x_1 \| \leqslant \varepsilon^\varkappa$ imply that

$$\| \bar{x}_1 \| \leqslant \left\{ 2\delta_2 \varepsilon^{\zeta - \varkappa} \left[\frac{1}{d_1} (\| \Delta_1 \| \varepsilon^{\varkappa - 1} + \| \Delta_2 \|) \right] \right\} \varepsilon^\varkappa$$

It is obvious that for small ε the expression in braces is smaller than unity, i.e. $T(K_\varepsilon) \subset K_\varepsilon$.

2°. We shall prove the last of estimates (4.2) by setting $\| x_1 \| \leqslant \varepsilon^\varkappa$ and noting that

$$\left\| \frac{\partial x_0}{\partial \theta_1} \right\| + \left\| \frac{\partial y_0}{\partial \theta_1} \right\| + \left| \frac{\partial \theta_0}{\partial \theta_1} - 1 \right| < M_1 \varepsilon, \quad M_1 = \text{const} \qquad (4.3)$$

Evidently

$$\left| \frac{\partial \bar{\theta}_1}{\partial \theta_1} \right| \geqslant \left| \frac{\partial \bar{\theta}_1}{\partial t_n} \frac{\partial t_n}{\partial \theta_1} \right| - \left\| \frac{\partial \bar{\theta}_1}{\partial x_0} \right\| \left\| \frac{\partial x_0}{\partial \theta_1} \right\| - \left\| \frac{\partial \bar{\theta}_1}{\partial y_0} \right\| \left\| \frac{\partial y_0}{\partial \theta_1} \right\| - \left\| \frac{\partial \bar{\theta}_1}{\partial \theta_0} \right\| \left\| \frac{\partial \theta_0}{\partial \theta_1} \right\|$$

Using (4.1) and (4.2) we obtain

$$\left| \frac{\partial \bar{\theta}_1}{\partial t_n} \frac{\partial t_n}{\partial \theta_1} \right| \geqslant \frac{7}{8} \frac{\omega}{\gamma_1} \left| \frac{\Delta_{2\theta_1}(0, \theta_1, 0)}{\Delta_2(0, \theta_1, 0)} \right| + \alpha_1(\varepsilon)$$

here and below $\alpha_i \to 0$ when $\varepsilon \to 0$.

From (3.7) and the equality $y_0 e^{\gamma(\varepsilon, k\varepsilon) t_n} = d_1$ follows that

$$\left\| \frac{\partial \bar{\theta}_1}{\partial y_0} \right\| \left\| \frac{\partial y_0}{\partial \theta_1} \right\| \leqslant C_2' e^{\gamma(\varepsilon, k\varepsilon) t_n} | y_{0\theta_1} | = C_2' d_1 \left| \frac{y_{0\theta_1}}{y_0} \right|$$
$$< \frac{\omega}{8\gamma_1} \left| \frac{\Delta_{2\theta_1}(0, \theta_1, 0)}{\Delta_2(0, \theta_1, 0)} \right| + \alpha_2(\varepsilon)$$

(the quantity δ_3 and consequently, also, d_1 is selected so small that $C_2 d_1 < \omega/(8\gamma_1)$).

It follows from (3.6) and (4.3) that

$$\left\|\frac{\partial\bar\theta_1}{\partial x_0}\right\|\left\|\frac{\partial x_0}{\partial\theta_1}\right\| \leqslant C_1' e^{N\delta t_n} \leqslant C_1' M_1 \left|\frac{y_0}{d_1}\right|^{-N\delta/\gamma_1} < \mathrm{const}\, \varepsilon^{1-N\delta/\gamma_1} = \alpha_3(\varepsilon)$$

(the quantity δ_3 is selected so small that $N\delta_3 < \gamma_1$).

From (4.3) and (3.8)

$$\left|\frac{\partial\bar\theta_1}{\partial\theta_0}\frac{\partial\theta_0}{\partial\theta_1}\right| \leqslant \frac{9}{8} + \alpha_4(\varepsilon)$$

The above inequalities yield the last of estimates (4.2).

3°. From (3.15) it is possible to derive

$$\left\|\frac{\partial\theta_0}{\partial x_1}\right\| + \left\|\frac{\partial x_0}{\partial x_1}\right\| + \left\|\frac{\partial y_0}{\partial x_1}\right\| \leqslant M_2 = \mathrm{const} \tag{4.4}$$

Obviously

$$\left\|\frac{\partial\theta_1}{\partial x_1}\right\| \leqslant \left\|\frac{\partial\bar\theta_1}{\partial t_n}\right\|\left\|\frac{\partial t_n}{\partial x_1}\right\| + \left\|\frac{\partial\bar\theta_1}{\partial(x_0, y_0, \theta_0)}\right\|\left\|\frac{\partial(x_0, y_0, \theta_0)}{\partial x_1}\right\| \tag{4.5}$$

From (4.1) and (3.6), (4.4) and (3.10), and (4.4) and (3.11) we have, respectively,

$$\left\|\frac{\partial t_n}{\partial x_1}\right\| \leqslant \mathrm{const}\,\varepsilon^{-1} + \alpha_6(\varepsilon)$$

$$\left\|\frac{\partial\theta_1}{\partial x_0}\right\|\left\|\frac{\partial x_0}{\partial x_1}\right\| \leqslant C_2' M_2 e^{N\delta t_n} \leqslant C_2' M_2 e^{\gamma(\varepsilon, k\varepsilon)t_n} = C_2 M_2 d_1 |y_0|^{-1}$$

$$\leqslant \mathrm{const}\,\varepsilon^{-1} + \alpha_7(\varepsilon), \quad \left\|\frac{\partial\bar\theta_1}{\partial y_0}\right\|\left\|\frac{\partial y_0}{\partial x_1}\right\| \leqslant C_2' e^{\gamma(\varepsilon, k\varepsilon)t_n} \leqslant \mathrm{const}\,\varepsilon^{-1} + \alpha_8(\varepsilon)$$

$$\left\|\frac{\partial\bar\theta_1}{\partial\theta_0}\right\|\left\|\frac{\partial\theta_0}{\partial x_1}\right\| \leqslant \mathrm{const}$$

which yields the third of estimates (4.2). Then from (4.4), (3.10), (3.6)-(3.8) follows

$$\left\|\frac{\partial\bar x_1}{\partial x_1}\right\| \leqslant L_1\varepsilon^{\zeta-1} + L_2\varepsilon^{-\beta/\gamma_1} + L_3\varepsilon^{(\zeta-1)} + L_4\varepsilon^{\zeta-\alpha/\gamma_1} \tag{4.6}$$

where L_i are constants, and each term in (4.6) is the estimate or the corresponding term in the right-hand side of an inequality similar to (4.5) in which $\bar x_1$ has been substituted for θ_1. From (4.3) and (3.9) we obtain the first of estimates (4.2).

Finally, from (4.3), (4.1), (3.10), and (3.6)-(3.8) follows that

$$\left\|\frac{\partial\bar x_1}{\partial\theta_1}\right\| \leqslant L_1\varepsilon^{\zeta} + L_2'\varepsilon^{\zeta-1} + L_3\varepsilon^{-\beta/\gamma_1} + L_4\varepsilon^{\zeta-\alpha/\gamma_1} \tag{4.7}$$

From (4.7) and (3.9) we obtain the second of estimates (4.2). The lemma is proved.

In Fig. 3 ring K_ε and its image in mapping T_1 for $\varepsilon > 0$ are shown hatched.

Theorem. Let us assume the existence of interval $I=[a, b]$, $0 < a < b \leqslant 2\pi$ such that for $\theta_1 \in I$

$$\frac{\omega}{\gamma_1}\left|\frac{\Delta_{2\theta_1}(0,\theta_1,0)}{\Delta_2(0,\theta_1,0)}\right| > 3 \tag{4.8}$$
$$\frac{\omega}{\gamma_1}\left|\ln\frac{\Delta_2(0,b,0)}{\Delta_2(0,a,0)}\right| \geqslant 3\pi n, \quad n \geqslant 3 \tag{4.9}$$

Then for all reasonably small ε the mapping T has in the ring K_ε an invariant set on which T is associated with the shift of the Bernoulli scheme of $n \dotplus 1$ symbols.

Proof. Let us check conditions of the ring principle. It follows from (4.2) and (4.7) that $|\partial\bar{\theta}_1/\partial\theta_1| > 9/8$ for $\theta_1 \in I$ and, furthermore, $\|\partial\bar{x}_1/\partial x_1\| < 1$. We now set

$$t_b = -\frac{1}{\gamma(\varepsilon, k\varepsilon)}\ln\frac{y_0(x_1, b, \varepsilon)}{d_1}, \quad y_b = d_1 e^{-\gamma(\varepsilon, k\varepsilon)t_b}$$
$$t_a = -\frac{1}{\gamma(\varepsilon, k\varepsilon)}\ln\frac{y_0(x_1, a, \varepsilon)}{d_1}, \quad y_a = d_1 e^{-\gamma(\varepsilon, k\varepsilon)t_a}$$

Then

$$|\bar{\theta}_1(b) - \bar{\theta}_1(a)| > \left|\frac{\partial\bar{\theta}_1}{\partial t_n}\right| |t_b - t_a| - \left\|\frac{\partial\bar{\theta}_1}{\partial x_0}\right\| \left\|\frac{\partial x_0}{\partial\theta_1}\right\| |b-a|$$
$$-\left\|\frac{\partial\bar{\theta}_1}{\partial y_0}\right\| \left\|\frac{\partial y_0}{\partial t}\right\| |t_b - t_a| - \left\|\frac{\partial\bar{\theta}_1}{\partial\theta_0}\right\| \left\|\frac{\partial\theta_0}{\partial\theta_1}\right\| |b-a|$$

Using the inequalities

$$|t_b - t_a| > \frac{1}{\gamma_1}\left|\ln\frac{\Delta_2(0,b,0)}{\Delta_2(0,a,0)}\right| + \alpha_9(\varepsilon)$$
$$\left|\frac{\partial\bar{\theta}_1}{\partial y_0}\right| \left|\frac{\partial y_0}{\partial t}\right| \leqslant \gamma_1 C_2' d_1$$

the second of which follows from (3.7), and taking into account (3.6) and the inequalities (4.3), (3.10), and (3.11) we obtain

$$|\bar{\theta}_1(b) - \theta_1(a)| > \frac{\omega}{\gamma_1}\left|\ln\frac{\Delta_2(0,b,0)}{\Delta_2(0,a,0)}\right| \left|\frac{7}{8} - \gamma_1 C_2' d_1\right| - \frac{9}{8}$$
$$+ \alpha_{10}(\varepsilon) > \frac{6\omega}{8\gamma_1}\left|\ln\frac{\Delta_2(0,b,0)}{\Delta_2(0,a,0)}\right| - \frac{9}{8}$$

(the quantity δ_3 is selected so small that $\gamma_1 C_2'\delta_3 < 1/8$). From this and (4.9) we obtain, $\bar{\theta}_1(b) - \bar{\theta}_1(a)\,| > 2\pi n$. For the final proof of conditions of the ring principle it is sufficient to show that

$$\left\|\frac{\partial\bar{\theta}_1}{\partial x_1}\right\| \left\|\frac{\partial x_1}{\partial\theta_1}\right\| < \frac{1}{8}$$

It follows from (4.2) and (3.9) that $D_2 D_3 \varepsilon^{\zeta - \alpha/\gamma_1}\varepsilon^{-1} < 1/8$ when ε is small.

Note 2. Function $\Delta_2(0, \theta, 0)$ defines the slit between the traces of manifolds W_ε^+ and W_ε^- on secant S_0. If the derivative of that function is small in comparison with the function itself (e.g., $\Delta_2 = 3 + \sin\omega_0\theta$,

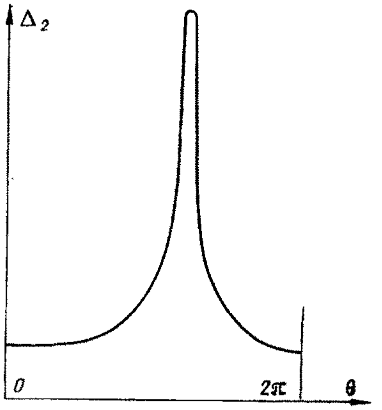

Fig. 4

$\omega_0 \ll 1$), the mapping T has an invariant curve in K_ε. If there exists a segment on which the derivative is very large, for instance when Δ_2 is of the pulse kind (Fig. 4), we are faced with the ring principle, and Fig. 4 corresponds exactly to mapping T of the form shown in Fig. 1. Between these two limit cases lies the intermediate case associated with the Smiles horseshoe (see [9].

REFERENCES

1. Krylov, N.M. and Bogoliubov, N.N., Application of methods of nonlinear mechanics in the theory of steady oscillations. Kiev, Izd. Vseukr. Akad. Nauk, 1934.
2. Afraimovich, V.S. and Shil'nikov, L.P., On small periodic perturbations of autonomous systems. Dokl. Akad. Nauk SSSR, Vol. 214, №4, 1974.
3. Afraimovich, V.S. and Shil'nikov, L.P., On certain combined bifurcations associated with the disappearance of a stationary point of the saddle-node kind. Dokl. Akad. Nauk SSSR, Vol. 219, №6, 1974.
4. Shil'nikov, L.P., Theory of dynamic system bifurcation and dangerous boundaries. Dokl. Akad. Nauk SSSR, Vol. 224, №5, 1975.
5. Andronov, A.A., Leontovich, E.A., Gordon, I.I., and Maier, A.G., The Theory of Bifurcation of Dynamic Systems in a Plane. Moscow, "Nauka", 1967.
6. Shil'nikov, L.P., On certain cases of periodic motion creation from singular trajectories. Matem. Sb., Vol. 61, №4, 1963.
7. Kelley, A., The stable, center-stable, center, center-unstable, and unstable manifolds; an appendix in: R. Abraham and J. Robbin. Transversal Mappings and Flows. New York, Benjamin, 1967.
8. Shil'nikov, L.P., On a particular Poincaré-Birkhoff problem. Matem. Sb., Vol. 74, №3, 1967.
9. Gavrilov, N.K. and Shil'nikov, L.P., On three-dimensional dynamic systems with a noncoarse homicline curve. I. Matem. Sb., Vol. 88, №4, 1972, and II. Matem. Sb., Vol. 90, №1, 1973.

Translated by J.J. D.